

Tilburg University

## A simplicial algorithm for the nonlinear stationary point problem on an unbounded polyhedron

Talman, A.J.J.; Dai, Y.; van der Laan, G.; Yamamoto, Y.

*Published in:*  
SIAM Journal on Optimization

*Publication date:*  
1991

[Link to publication in Tilburg University Research Portal](#)

### *Citation for published version (APA):*

Talman, A. J. J., Dai, Y., van der Laan, G., & Yamamoto, Y. (1991). A simplicial algorithm for the nonlinear stationary point problem on an unbounded polyhedron. *SIAM Journal on Optimization*, 1(1), 151-165.

### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

### Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.



## A SIMPLICIAL ALGORITHM FOR THE NONLINEAR STATIONARY POINT PROBLEM ON AN UNBOUNDED POLYHEDRON\*

Y. DAI†, G. VAN DER LAAN‡, A. J. J. TALMAN§, AND Y. YAMAMOTO¶¶

**Abstract.** A path-following algorithm is proposed for finding a solution to the nonlinear stationary point problem on an unbounded, convex, and pointed polyhedron. The algorithm can start at an arbitrary point of the polyhedron. The path to be followed by the algorithm is described as the path of zeros of some piecewise continuously differentiable function defined on an appropriate subdivided manifold. This manifold is induced by a generalized primal-dual pair of subdivided manifolds. The path of zeros can be approximately followed by dividing the polyhedron into simplices and replacing the original function by its piecewise linear approximation with respect to this subdivision. The piecewise linear path of this function can be generated by alternating replacement steps and linear programming pivot steps. A condition under which the path of zeros converges to a solution is also stated, and a description of how the algorithm operates when the problem is linear or when the polyhedron is the Cartesian product of a polytope and an unbounded polyhedron is given.

**Key words.** simplicial algorithm, stationary point problem, subdivided manifold, convergence condition

**AMS(MOS) subject classifications.** 90A14, 90A30, 90A33

**1. Introduction.** Let  $K$  be a convex polyhedron in  $R^n$ . We assume that  $K$  is unbounded and pointed, i.e.,  $K$  has at least one vertex, and that  $K$  is represented by the set  $\{x \in R^n \mid A'x \leq b\} = \{x \in R^n \mid (a_i)'x \leq b_i \text{ for } i = 1, \dots, m\}$ , where  $A$  is an  $n \times m$  matrix consisting of column vectors  $a_i$  for  $i = 1, \dots, m$ , and  $b = (b_1, \dots, b_m)'$  is an  $m$ -vector. Further, let  $f$  be a continuously differentiable function from  $K$  to  $R^n$ . Then the (nonlinear) *stationary point problem* for  $f$  on  $K$  is to find a point  $x$  in  $K$  such that

$$(z - x)'f(x) \geq 0$$

for any point  $z$  in  $K$ . We call  $x$  a *stationary point* of  $f$  on  $K$ . If the function  $f$  is affine on  $K$ , we call the problem the *linear stationary point problem*. The stationary point problem on an unbounded convex polyhedron is frequently met in mathematical programming, for example, to find a Karush–Kuhn–Tucker point for an optimization problem with linear constraints.

To solve the nonlinear stationary point problem on  $K$  we propose a path-following algorithm. Such an algorithm traces the set of zeros of a piecewise continuously differentiable function  $g$  defined from an  $(n+1)$ -dimensional subdivided manifold to  $R^n$ . In case the zero vector is a regular value of the function  $g$  there exists a path of zeros initiating from an arbitrarily chosen point in  $K$ . The  $(n+1)$ -dimensional subdivided manifold is induced by a generalized primal-dual pair of subdivided manifolds, where the primal sets are determined by the faces of  $K$  and the dual sets are determined by the normal cones of these faces. A primal-dual pair of subdivided manifolds is a basic framework in path-following techniques for finding fixed points or solving stationary point problems (see, for example, [1], [2], [4], [7]–[9], and [10]).

\* Received by the editors October 30, 1989; accepted for publication (in revised form) August 16, 1990.

† Institute of Socio-Economic Planning, University of Tsukuba, Tsukuba, Ibaraki 305, Japan.

‡ Department of Econometrics, Vrije Universiteit, P.O. Box 7161, 1007 MC Amsterdam, the Netherlands.

§ Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, the Netherlands.

¶ The research of this author was partly supported by Alexander von Humboldt-Stiftung while he visited Forschungsinstitut für Diskrete Mathematik/Institut für Ökonometrie und Operations Research, Universität Bonn, Federal Republic of Germany.



The path  $S$  of zeros of the function can be approximately followed by a simplicial algorithm. This algorithm subdivides first the set  $K$  into simplices in some appropriate way and replaces the function  $f$  by its piecewise linear approximation  $\bar{f}$  with respect to this triangulation. For this function the path of zeros of  $g$  becomes piecewise linear and can therefore be followed by making alternating replacement steps and linear programming pivot steps for a sequence of adjacent simplices of varying dimension.

Since the set  $K$  is unbounded, the path  $S$  may diverge to infinity. We state a simple condition on the function under which the path  $S$  is bounded and therefore leads from the starting point to a solution of the problem. We also describe how the algorithm should be adapted in case  $K$  is the Cartesian product of a polytope, i.e., a bounded convex polyhedron, and an unbounded convex polyhedron, and under what condition the path  $S$  is bounded for this case. We conclude the paper with a short description of the algorithm when the function is affine on  $K$ . The convergence condition for this problem is related to the well-known condition of copositive plus in case of the linear complementarity problem.

This paper is a generalization of path-following techniques introduced earlier for solving stationary point problems. In [8] such a method has been proposed for the linear stationary point problem on a polytope. In [7] the nonlinear stationary point problem on a polytope was treated. Finally, in [1] a path-following algorithm for the linear stationary point problem on a polyhedral cone was introduced.

This paper is organized as follows. Section 2 briefly reviews a basic theorem for path-following algorithms and extends the concept of a primal-dual pair of subdivided manifolds. In § 3 we describe the generalized pair of primal-dual subdivided manifolds which will underlie the algorithm. Section 4 defines the path of zeros leading from an arbitrary point to either infinity or a solution. We describe how this path can be approximately followed by a simplicial algorithm. In § 5 we state a convergence condition guaranteeing that the path is bounded. Finally, §§ 6 and 7 discuss the cases when  $K$  is the product of a polytope and a convex polyhedron and when  $f$  is affine on  $K$ , respectively.

**2. Generalization of the primal-dual pair of subdivided manifolds.** We shall briefly review a basic theorem for path-following algorithms and extend the concept of the primal-dual pair of subdivided manifolds introduced by Kojima and Yamamoto [4].

We call an  $l$ -dimensional convex polyhedron a *cell* or an  *$l$ -cell*. When a cell  $X$  is a face (see, for example, [6]) of a cell  $Y$ , we write  $X \leq Y$ . We denote  $X < Y$  when  $X$  is a proper face of  $Y$ . Particularly when an  $(l-1)$ -cell  $X$  is a face of an  $l$ -cell  $Y$ , we call  $X$  a *facet* of  $Y$  and denote it by  $X \triangleleft Y$ .

A collection  $\mathcal{L}$  of cells of the same dimension, say  $l$ , is called an  $l$ -dimensional subdivided manifold if it satisfies the following conditions:

- (1) any two cells of  $\mathcal{L}$  intersect in a common face unless the intersection is empty,
- (2) any facet of a cell of  $\mathcal{L}$  lies in at most two cells of  $\mathcal{L}$ ,
- (3) each point of cells of  $\mathcal{L}$  has a neighborhood which intersects finitely many cells of  $\mathcal{L}$ .

We denote the collection of all faces of cells of  $\mathcal{L}$  by  $\bar{\mathcal{L}}$ , i.e.,

$$\bar{\mathcal{L}} = \{X \mid X \text{ is a face of some cell of } \mathcal{L}\},$$

and the union of all cells of  $\mathcal{L}$  by  $|\mathcal{L}|$ , i.e.,

$$|\mathcal{L}| = \bigcup [X \mid X \text{ is a cell of } \mathcal{L}].$$

It is noteworthy that  $\bar{\mathcal{L}}$  consists of cells of various dimensions. By the second and most crucial condition, each  $(l-1)$ -cell of  $\bar{\mathcal{L}}$  lies in either one or two  $l$ -cells of  $\mathcal{L}$ . We



refer to the collection of those  $(l-1)$ -cells lying in exactly one  $l$ -cell of  $\mathcal{L}$  as the *boundary* of  $\mathcal{L}$  and denote it by  $\partial\mathcal{L}$ . A continuous mapping  $g$  from  $|\mathcal{L}|$  into  $R^n$  is *piecewise continuously differentiable* (abbreviated by  $PC^1$ ) on  $\mathcal{L}$  if the restriction of  $g$  to each cell of  $\mathcal{L}$  has a continuously differentiable extension. We denote the Jacobian matrix of  $g$  at a point  $x$  of a cell  $C$  of  $\mathcal{L}$  by  $Dg(x; C)$ . A point  $c \in R^n$  is a *regular value* of the  $PC^1$  mapping  $g: |\mathcal{L}| \rightarrow R^n$  if

$$x \in B \leq C \in \mathcal{L} \quad \text{and} \quad g(x) = c \quad \text{imply} \quad \dim \{Dg(x; C)y \mid y \in B\} = n.$$

We now state one of the basic theorems for a path-following algorithm [2].

**THEOREM 2.1.** *Let  $\mathcal{L}$  be an  $(n+1)$ -dimensional subdivided manifold in some Euclidean space and let  $g: |\mathcal{L}| \rightarrow R^n$  be a  $PC^1$  mapping. Suppose  $c \in R^n$  is a regular value of  $g$  and  $g^{-1}(c) \neq \emptyset$ . Then  $g^{-1}(c)$  is a disjoint union of paths and loops, where a path is a connected one-dimensional manifold homeomorphic to one of the intervals  $(0, 1)$ ,  $(0, 1]$ , and  $[0, 1]$ , and a loop is a connected one-dimensional manifold homeomorphic to the one-dimensional sphere. Furthermore,  $g^{-1}(c)$  has the following properties:*

- (1)  $g^{-1}(c) \cap X$  is either empty or a disjoint union of smooth one-manifolds for each  $X \in \mathcal{L}$ ,
- (2) a loop of  $g^{-1}(c)$  does not intersect  $|\partial\mathcal{L}|$ ,
- (3) if a path  $S$  of  $g^{-1}(c)$  is compact, the boundary  $\partial S$  of  $S$  consists of two distinct points in  $|\partial\mathcal{L}|$ .

We first generalize the primal-dual pair of subdivided manifolds proposed in [4]. In [4] the dual operator relating a pair of subdivided manifolds was assumed to satisfy several conditions including one-to-one. We will here relax these conditions. Let  $\mathcal{P}$  and  $\mathcal{D}$  be subdivided manifolds. A dual operator, say  $d$ , is defined on  $\bar{\mathcal{P}}$  and assigns to each cell of  $\bar{\mathcal{P}}$  either the empty set or a cell  $Y$  of  $\bar{\mathcal{D}}$  such that for some fixed positive integer  $l$ , called the degree,

$$\dim X + \dim Y = l$$

holds. We denote the image of  $X \in \bar{\mathcal{P}}$  under the operator  $d$  by  $X^d$ . When a pair of subdivided manifolds  $\mathcal{P}$  and  $\mathcal{D}$  is linked by such an operator  $d$ , we call the triplet  $(\mathcal{P}, \mathcal{D}; d)$  a *generalized primal-dual pair of subdivided manifolds*, GPDM for short. We allow a dual operator to assign the same cell of  $\bar{\mathcal{D}}$  to more than one cell of  $\bar{\mathcal{P}}$ , that is, to be a noninjective dual operator. Letting

$$(2.1) \quad \mathcal{L} = \{X \times X^d \mid X \in \bar{\mathcal{P}}, X^d \neq \emptyset\},$$

the conditions required for  $\mathcal{L}$  to be a subdivided manifold are given in the next lemma.

**LEMMA 2.2.** *Suppose  $(\mathcal{P}, \mathcal{D}; d)$  is a GPDM with degree  $l$ . Let  $\mathcal{L}$  be defined by (2.1). Then  $\mathcal{L}$  is an  $l$ -dimensional subdivided manifold if and only if for every  $(l-1)$ -cell  $X \times Y$  of  $\bar{\mathcal{L}}$ :*

- (1) *there are at most two cells  $Z$  of  $\bar{\mathcal{P}}$  such that*

$$(2.2) \quad X \triangleleft Z \quad \text{and} \quad Z^d = Y,$$

- (2) *if  $Y \triangleleft X^d$ , then there is at most one cell  $Z$  of  $\bar{\mathcal{P}}$  satisfying (2.2).*

*Proof.* Among the three conditions of a subdivided manifold the second one is crucial and the others will be straightforward. Note that an  $(l-1)$ -cell  $X \times Y$  of  $\bar{\mathcal{L}}$  is a facet of an  $l$ -cell  $Z \times Z^d$  of  $\mathcal{L}$  if and only if either

$$(2.3) \quad X = Z \quad \text{and} \quad Y \triangleleft Z^d$$

or

$$(2.4) \quad X \triangleleft Z \quad \text{and} \quad Y = Z^d$$



holds. From the first condition (1) it follows that there are at most two cells  $Z \times Z^d$  of  $\mathcal{L}$  satisfying (2.4). Furthermore, condition (2) means that there is at most one such cell if a cell  $Z \times Z^d = X \times X^d$  satisfying (2.3) exists. Therefore we have shown that  $X \times Y$  lies in at most two  $l$ -cells of  $\mathcal{L}$ .

The “only if” part is readily seen by the same argument.  $\square$

The following lemma characterizes the cells constituting the boundary  $\partial\mathcal{L}$  of  $\mathcal{L}$ .

**LEMMA 2.3.** *An  $(l-1)$ -cell  $X \times Y$  of  $\bar{\mathcal{L}}$  belongs to the boundary  $\partial\mathcal{L}$  of  $\mathcal{L}$  if and only if the following conditions hold:*

- (1) *if  $Y \triangleleft X^d$ , then there is no cell  $Z$  of  $\bar{\mathcal{P}}$  satisfying (2.2),*
- (2) *if  $Y \not\triangleleft X^d$ , then there is exactly one cell  $Z$  of  $\bar{\mathcal{P}}$  satisfying (2.2).*

**3. Construction of a GPDM.** In what follows, we shall present a subdivision of the polyhedron  $K$  and construct a GPDM having this subdivision as the primal subdivided manifold. We assume that  $K$  is unbounded.

It is well known that  $K$  can be decomposed into a polytope and a polyhedral cone  $C$  containing the directions of all rays in  $K$ , and that  $C$  is given by  $C = \{x \mid A'x \leq 0\}$  (see, for example, [6]). Since  $K$  is pointed, the cone  $C$  of rays is also pointed, namely,  $C \cap (-C) = \{0\}$ . Indeed, suppose that  $r \in C \cap (-C)$  and consider two points  $v + r$  and  $v - r$  for an arbitrarily chosen vertex  $v$  of  $K$ . Since  $r$  and  $-r \in C$ , both of these two points lie in  $K$ . If  $r \neq 0$ , then the point  $v$  would be a middle point of these points, which contradicts the fact that  $v$  is a vertex.

Let  $w$  be an arbitrary point of  $K$ . In the algorithm proposed below for solving the stationary point problem on  $K$  the point  $w$  will be the starting point. For some strictly positive  $m$ -vector  $\gamma$ , let  $h = -A\gamma$  and let  $H^0 = \{x \mid h'x = h_0\}$  be a hyperplane for some positive number  $h_0$ . We can see that if  $h_0$  is sufficiently large, this hyperplane intersects every unbounded face of  $K$  while the negative halfspace  $H^- = \{x \mid h'x \leq h_0\}$  contains  $w$  and all vertices of  $K$  in its interior and hence all bounded faces of  $K$ . To see this, let  $r$  be a nonzero vector of  $C$ . Since  $C$  is pointed,  $A'r \neq 0$ . More precisely,  $A'r \leq 0$  and  $(a_i)'r < 0$  for at least one column  $a_i$  of  $A$ . Then, by the definition of  $h$ ,

$$(3.1) \quad h'r = -\gamma'A'r > 0.$$

Therefore, when  $h_0$  is large enough for the interior of  $H^-$  to contain all vertices of  $K$  and  $w$ , the hyperplane  $H^0$  intersects every unbounded face of  $K$ .

Now we introduce several notations. Let  $H^+ = \{x \mid h'x \geq h_0\}$  be the positive halfspace of  $H^0$ . For any face  $F$  of  $K$ , let

$$F^- = \{x \mid x \in F \cap H^-\},$$

$$F^0 = \{x \mid x \in F \cap H^0\},$$

and

$$F^+ = \{x \mid x \in F \cap H^+\}.$$

Note that if some face  $F$  is entirely included in  $H^-$ , then  $F^- = F$ . For an arbitrary subset  $G$  of  $K$ , we denote the convex hull of  $G$  and  $w$  by  $wG$ . Let

$$(3.2) \quad \mathcal{P} = \{wF^- \mid w \notin F \triangleleft K\} \cup \{wK^0\} \cup \{K^+\}.$$



It can easily be proved that  $\mathcal{P}$  is a subdivided manifold with the same dimension as  $K$ . Moreover, the collection  $\bar{\mathcal{P}}$  is equal to

$$\begin{aligned}
 \bar{\mathcal{P}} = & \{wF^- \mid w \notin F < K\} \\
 & \cup \{wF^0 \mid F \text{ is an unbounded face of } K\} \\
 & \cup \{F^+ \mid F \text{ is an unbounded face of } K\} \\
 & \cup \{F^0 \mid F \text{ is an unbounded face of } K\} \\
 & \cup \{F^- \mid w \notin F < K\} \\
 & \cup \{w\},
 \end{aligned}
 \tag{3.3}$$

and

$$|\mathcal{P}| = K. \tag{3.4}$$

An example is illustrated in Fig. 3.1.

To make the dual subdivided manifold  $\mathcal{D}$ , we subdivide  $R^n$  in almost the same way as in [1]. The *normal cone* at  $x \in K$  to  $K$  is defined to be

$$N(x, K) = \{y \mid y'(z - x) \leq 0 \text{ for every } z \in K\}. \tag{3.5}$$

It is the cone of all outward normal vectors at  $x$  to  $K$ . It is readily seen that normal cones are identical at any relative interior point of a face  $F$  of  $K$ . Therefore we denote it by  $F^*$ . Letting

$$I(F) = \{i \mid (a_i)'x = b_i \text{ for every } x \in F\},$$

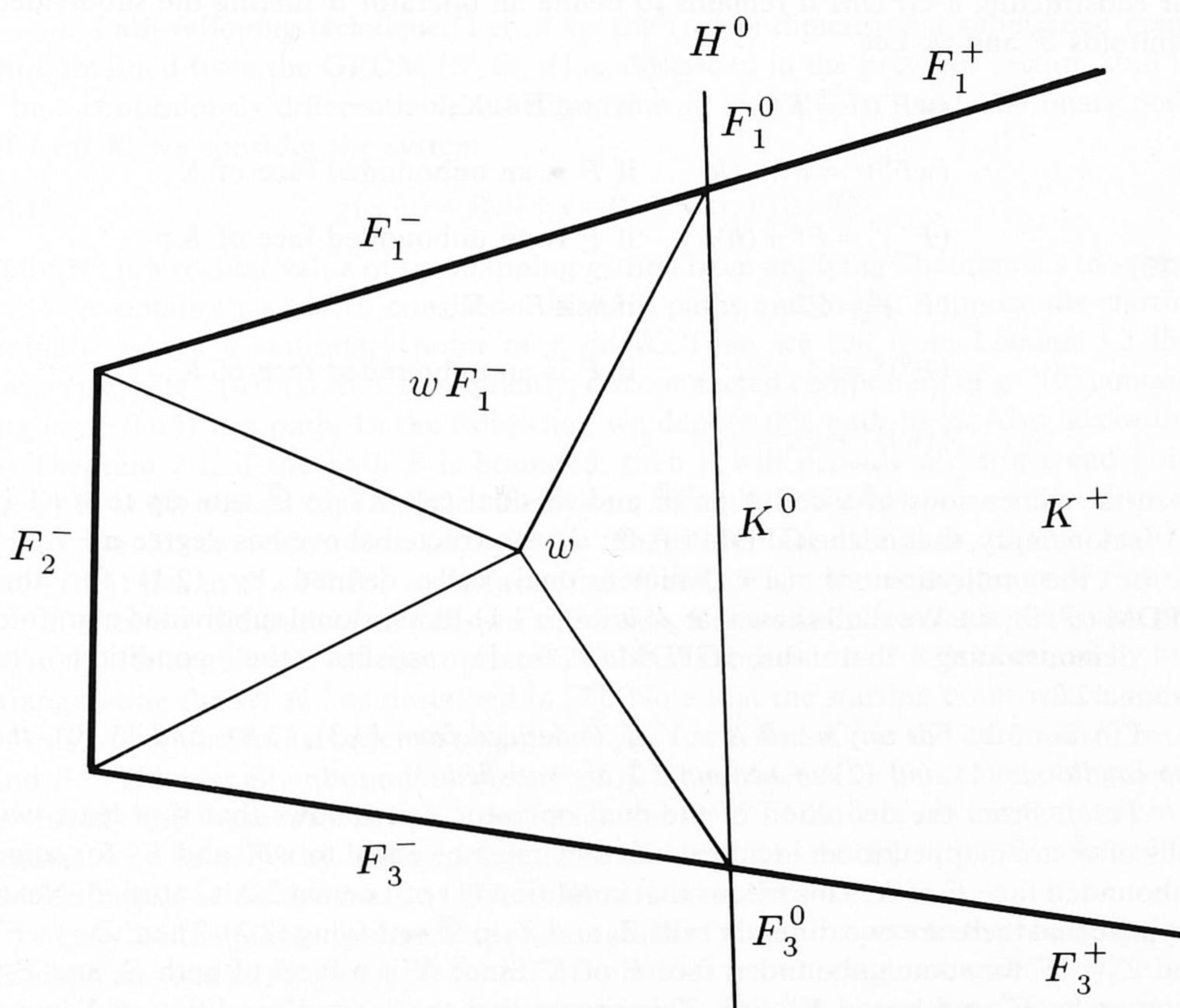


FIG. 3.1



then we obtain

$$F^* = \left\{ y \mid y = \sum_{i \in I(F)} \mu_i a_i, \mu_i \geq 0 \text{ for each } i \in I(F) \right\}.$$

The dual subdivided manifold  $\mathcal{D}$  is defined to be

$$(3.6) \quad \begin{aligned} \mathcal{D} = & \{ \{v\}^* \mid v \text{ is a vertex of } K \} \\ & \cup \{ F^* + \langle h \rangle \mid F \text{ is an extreme ray of } K \}, \end{aligned}$$

where

$$(3.7) \quad \langle h \rangle = \{ y \mid y = \alpha h \text{ for some } \alpha \geq 0 \},$$

being the ray in the direction  $h$ . Then  $\mathcal{D}$  is obviously an  $n$ -dimensional subdivided manifold,

$$(3.8) \quad \begin{aligned} \bar{\mathcal{D}} = & \{ F^* \mid F \leq K \} \\ & \cup \{ F^* + \langle h \rangle \mid F \text{ is an unbounded face of } K \}, \end{aligned}$$

and

$$(3.9) \quad |\mathcal{D}| = R^n.$$

For constructing a GPDM it remains to define an operator  $d$  linking the subdivided manifolds  $\mathcal{P}$  and  $\mathcal{D}$ . Let

$$(3.10) \quad \begin{aligned} (wF^-)^d &= F^* && \text{if } w \notin F < K; \\ (wF^0)^d &= F^* + \langle h \rangle && \text{if } F \text{ is an unbounded face of } K; \\ (F^+)^d &= F^* + \langle h \rangle && \text{if } F \text{ is an unbounded face of } K; \\ (F^-)^d &= \emptyset && \text{if } w \notin F < K; \\ (F^0)^d &= \emptyset && \text{if } F \text{ is an unbounded face of } K; \\ (\{w\})^d &= \emptyset. \end{aligned}$$

Then the dimensions of a cell  $X$  in  $\bar{\mathcal{P}}$  and its dual cell  $X^d$  in  $\bar{\mathcal{D}}$  sum up to  $n+1$  if  $X^d$  is nonempty, that is, the GPDM  $(\mathcal{P}, \mathcal{D}; d)$  constructed above has degree  $n+1$ . Let  $\mathcal{M}$  be the collection of  $(n+1)$ -dimensional cells defined by (2.1) for this GPDM  $(\mathcal{P}, \mathcal{D}; d)$ . We shall show that  $\mathcal{M}$  is an  $(n+1)$ -dimensional subdivided manifold by demonstrating that the GPDM  $(\mathcal{P}, \mathcal{D}; d)$  satisfies the conditions of Lemma 2.2.

LEMMA 3.1. *For any  $n$ -cell  $X \times Y$  of  $\bar{\mathcal{M}}$  derived from (3.3), (3.8), and (3.10), the two conditions (1) and (2) of Lemma 2.2 are satisfied.*

*Proof.* From the definition of the dual operator  $d$  it follows that if at least two cells of  $\bar{\mathcal{P}}$  are mapped to an identical cell they must be equal to  $wF^0$  and  $F^+$  for some unbounded face  $F$  of  $K$ . This means that condition (1) of Lemma 2.2 is satisfied. Next, suppose that there are two different cells  $Z_1$  and  $Z_2$  in  $\bar{\mathcal{P}}$  satisfying (2.2). Then,  $Z_1 = wF^0$  and  $Z_2 = F^+$  for some unbounded face  $F$  of  $K$ . Since  $X$  is a facet of both  $Z_1$  and  $Z_2$ ,  $X$  must be  $F^0$  and hence  $X^d = \emptyset$ . This proves that the second condition of Lemma 2.2 is also satisfied.  $\square$



Thus we have seen that  $\mathcal{M}$  is an  $(n+1)$ -dimensional subdivided manifold as an immediate consequence of Lemma 2.2. By applying Lemma 2.3 to the GPDM  $(\mathcal{P}, \mathcal{D}; d)$  considered here, we obtain the following lemma.

LEMMA 3.2.

$$\begin{aligned}
 \partial\mathcal{M} = & \{ \{w\} \times F^* \mid w \notin F < K, \dim F = 0 \} \\
 & \cup \{ \{w\} \times (F^* + \langle h \rangle) \mid F \text{ is an extreme ray of } K \} \\
 & \cup \{ F^+ \times F^* \mid F \text{ is an unbounded face of } K \} \\
 & \cup \{ F^- \times F^* \mid w \notin F < K \} \\
 & \cup \{ wF^0 \times F^* \mid w \in F, F \text{ is an unbounded face of } K \} \\
 & \cup \{ wF^- \times G^* \mid w \in G, w \notin F \triangleleft G, G \leq K \},
 \end{aligned}
 \tag{3.11}$$

and

$$\begin{aligned}
 |\partial\mathcal{M}| = & (\cup [\{w\} \times \{v\}^* \mid v \text{ is a vertex of } K, v \neq w]) \\
 & \cup (\cup [\{w\} \times (F^* + \langle h \rangle) \mid F \text{ is an extreme ray of } K]) \\
 & \cup (\cup [F \times F^* \mid \{w\} \neq F \leq K]).
 \end{aligned}
 \tag{3.12}$$

Note that

$$|\partial\mathcal{M}| \supset (\{w\} \times (R^n \setminus \{w\}^*)) \cup (\cup [F \times F^* \mid \{w\} \neq F \leq K]).$$

**4. Path-following technique.** Let  $\mathcal{M}$  be the  $(n+1)$ -dimensional subdivided manifold obtained from the GPDM  $(\mathcal{P}, \mathcal{D}; d)$  as described in the previous section, and let  $f$  be a continuously differentiable function from  $K$  into  $R^n$ . To find a stationary point of  $f$  on  $K$ , we consider the system

$$g(x, y) \equiv f(x) + y = 0, \quad (x, y) \in |\mathcal{M}|. \tag{4.1}$$

If  $0 \in R^n$  is a regular value of the mapping  $g$ , then from applying Theorem 2.1 to system (4.1) we obtain that  $g^{-1}(0)$  consists of disjoint paths and loops. Suppose the starting point  $w$  is not a stationary point of  $f$  on  $K$ . Then we see from Lemma 3.2 that  $(w, -f(w)) \in g^{-1}(0) \cap |\partial\mathcal{M}|$ . Consequently, the connected component of  $g^{-1}(0)$  containing  $(w, -f(w))$  is a path. In the following, we denote this path by  $S$ . Also, according to Theorem 2.1, if the path  $S$  is bounded, then it will provide a distinct end point  $(x, y)$  in  $|\partial\mathcal{M}|$ . Since  $(x, y)$  satisfies the system of equations (4.1),  $y = -f(x)$ . If  $x = w$ ,  $(x, y)$  would coincide with  $(w, -f(w))$ . Therefore, according to (3.12),  $(x, y) = (x, -f(x))$  lies in  $F \times F^*$  for some face  $F$  of  $K$  and  $x$  is a stationary point of  $f$  on  $K$ .

To follow the path  $S$  in  $|\mathcal{M}|$ , we subdivide  $K$  into simplices such that each cell  $X$  in  $\bar{\mathcal{P}}$  is triangulated. An appropriate simplicial subdivision of  $K$  is obtained by first triangulating the set  $K^-$  as described in [7]. Note that the starting point  $w$  is a vertex of this triangulation. In order to triangulate  $K^+$ , note that  $K^+$  is the union of  $K^0 + \langle h \rangle$  and  $F^+ + \langle h \rangle$  over all unbounded facets  $F$  of  $K$ . The subset  $K^0 + \langle h \rangle$  can be triangulated in exactly the same way as  $wK^0$ ; and each subset  $F^+ + \langle h \rangle$  can be triangulated in a similar way as  $wF^-$  by using projections of  $w + h$  on the faces of  $F^+$  instead of projections of  $w$  on the faces of  $F^-$ , as illustrated in Fig. 4.1.

Let  $\bar{f}$  be the piecewise linear approximation of  $f$  with respect to the triangulation. Taking  $\bar{f}$  instead of  $f$  in (4.1), the path  $T$  of solutions to (4.1) originating at  $(w, -f(w)) = (w, -\bar{f}(w))$  is piecewise linear and can therefore be followed by making pivoting steps in subsequent systems of linear equations. For ease of description we restrict ourselves



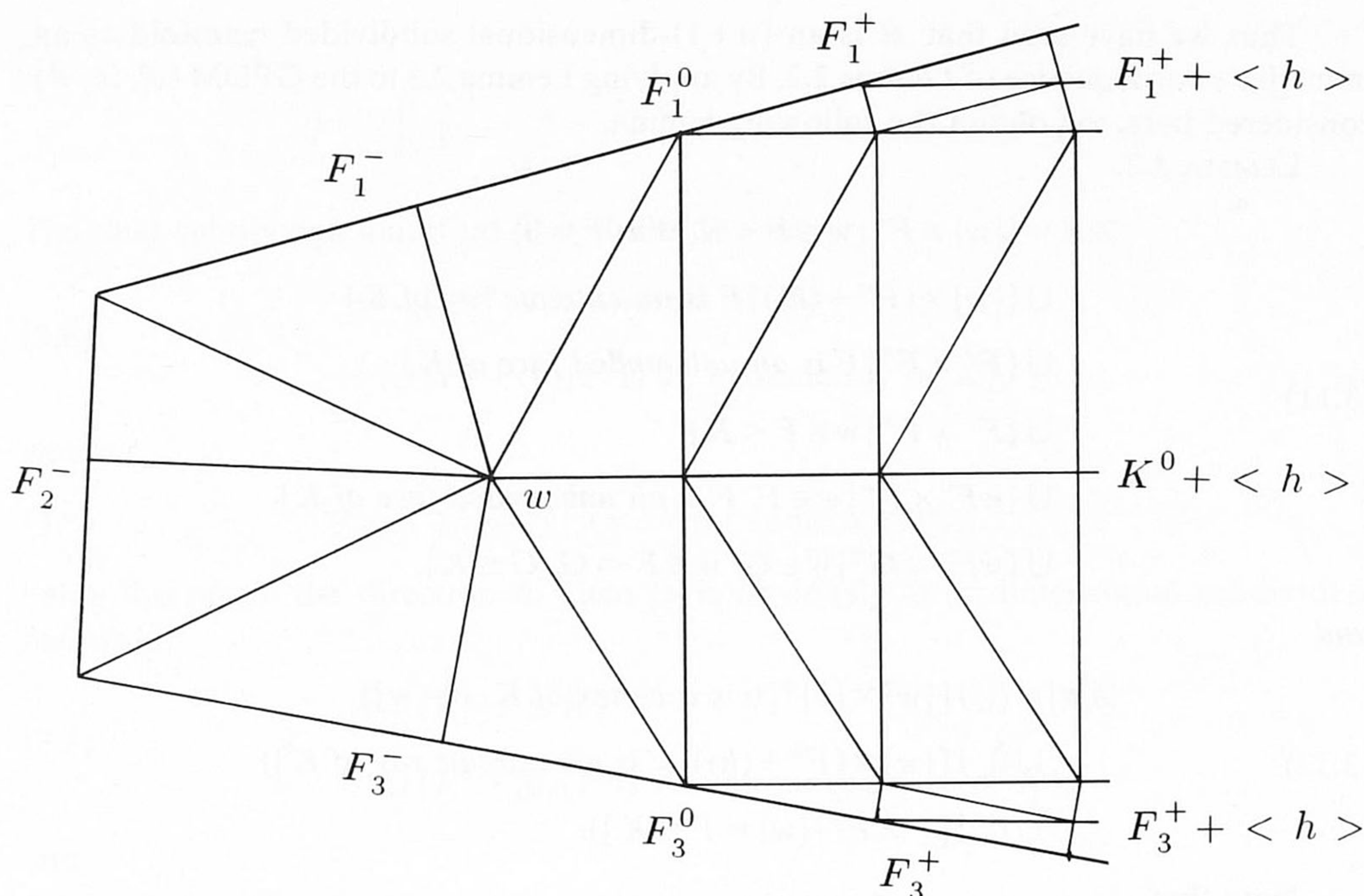


FIG. 4.1

to a polyhedron  $K$  for which none of the inequalities  $(a_i)'x \leq b_i$  is redundant and each vertex is an end point of exactly  $n$  one-faces of  $K$ . To start the algorithm we first solve the linear program

$$\begin{aligned}
 (4.2) \quad & \min \quad f(w)'x \\
 & \text{s.t.} \quad A'x \leq b \\
 & \quad \quad h'x \leq h_0.
 \end{aligned}$$

By the choice of  $h$  and  $h_0$ , this problem always has an optimal solution, which is some vertex, for instance  $v$ , of the feasible region. If the constraint  $h'x \leq h_0$  is not binding at  $v$ ,  $v$  is a vertex of  $K$  itself and so we take  $F = \{v\}$  and find a one-dimensional simplex  $\sigma$  of the triangulation in  $wF = wF^-$  which has  $w$  as a vertex. Let us denote  $w$  by  $w^1$  and the other vertex of  $\sigma$  by  $w^2$ . Let  $\mu_k^*$  be a dual optimal solution of (4.2) corresponding to the  $k$ th constraint  $(a_k)'x \leq b_k$ . Then, barring degeneracy  $\mu_k^* > 0$ , if and only if the  $k$ th constraint is binding at  $v$ , i.e.,  $k \in I(F)$ . Then we see that  $(\lambda_1^*, \lambda_2^*) = (1, 0)$  and  $\mu_k^*$ ,  $k \in I(F)$  satisfy the system of linear equations

$$\sum_{i=1}^2 \lambda_i \begin{bmatrix} f(w^i) \\ 1 \end{bmatrix} + \sum_{k \in I(F)} \mu_k \begin{bmatrix} a_k \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

When the constraint  $h'x \leq h_0$  is binding at  $v$ ,  $v$  lies on an extreme ray of  $K$ . We take the ray as  $F$  and find a one-dimensional simplex  $\sigma$  containing  $w$  as a vertex in  $wF^0$ . Let  $\mu_k^*$ ,  $k \in I(F)$ , and  $\alpha^*$  be a dual optimal solution of (4.2) corresponding to the  $k$ th constraint  $(a_k)'x \leq b_k$  and the last constraint  $h'x \leq h_0$ , respectively. Then we see that  $(\lambda_1^*, \lambda_2^*) = (1, 0)$ ,  $\mu_k^*$ ,  $k \in I(F)$ , and  $\alpha^*$  are a solution of

$$\sum_{i=1}^2 \lambda_i \begin{bmatrix} f(w^i) \\ 1 \end{bmatrix} + \sum_{k \in I(F)} \mu_k \begin{bmatrix} a_k \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



In both cases we leave the starting point  $w$  by increasing  $\lambda_2$  from zero. Note that  $(x, y)$ , which is  $(\sum_{i=1}^2 \lambda_i w^i, \sum_{k \in I(F)} \mu_k a_k)$  and  $(\sum_{i=1}^2 \lambda_i w^i, \sum_{k \in I(F)} \mu_k a_k + \alpha h)$ , respectively, is a point on the path  $T$  as long as all variables remain nonnegative.

Now, in general, let  $(x, y)$  be a point on the path  $T$ . Then in some  $t$ -cell  $X$  of  $\bar{\mathcal{P}}$  there is a simplex  $\sigma$  with vertices  $w^1, \dots, w^{t+1}$  such that  $x$  lies in  $\sigma$  and  $\bar{f}(x)$  in  $X^d$ . Hence, there exist nonnegative numbers  $\lambda_i^*, i = 1, \dots, t+1$ , such that  $x = \sum_i \lambda_i^* w^i$  and  $\sum_i \lambda_i^* = 1$ . Moreover, if  $X = wF^-$ , there exist nonnegative numbers  $\mu_k^*, k \in I(F)$ , such that  $y = \sum_{k \in I(F)} \mu_k^* a_k$ ; and if  $X = wF^0$  or  $X = F^+$ , there exist nonnegative numbers  $\mu_k^*, k \in I(F)$ , and  $\alpha^*$  such that  $y = \sum_{k \in I(F)} \mu_k^* a_k + \alpha^* h$ . In case not all vertices of  $K$  are determined by  $n$  one-faces of  $K$ , we refer to [7]. Since  $(x, y)$  is a solution of (4.1) with  $\bar{f}$  instead of  $f$  and  $\bar{f}(x) = \sum_i \lambda_i^* f(w^i)$ , it follows that  $\lambda_i^*, i = 1, \dots, t+1, \mu_k^*, k \in I(F)$ , is a nonnegative solution to the system of linear equations

$$(4.3) \quad \sum_{i=1}^{t+1} \lambda_i \begin{bmatrix} f(w^i) \\ 1 \end{bmatrix} + \sum_{k \in I(F)} \mu_k \begin{bmatrix} a_k \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

if  $X = wF^-$ , and that  $\lambda_i^*, i = 1, \dots, t+1, \mu_k^*, k \in I(F), \alpha^*$  is a nonnegative solution to the system of linear equations

$$(4.4) \quad \sum_{i=1}^{t+1} \lambda_i \begin{bmatrix} f(w^i) \\ 1 \end{bmatrix} + \sum_{k \in I(F)} \mu_k \begin{bmatrix} a_k \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

if  $X = wF^0$  or  $X = F^+$ . The system (4.3) or (4.4) has a line segment of solutions corresponding to a line segment of points  $x = \sum_i \lambda_i w^i$  in  $\sigma$ , assuming nondegeneracy. At an end point of solutions one of the variables is equal to zero. When  $\lambda_i = 0$  for some  $i \in \{1, \dots, t+1\}$ ,  $x$  lies in the facet  $\tau$  opposite the vertex  $w^i$  of  $\sigma$ . This facet lies either in the boundary of  $X$  or is a facet of just one other  $t$ -simplex  $\bar{\sigma}$  in the cell  $X$  with vertices  $w^j, j \neq i$ , and  $\bar{w}^i \neq w^i$ . Then in the latter case, to continue the path  $T$  in  $\bar{\sigma}$ , a pivoting step is made with  $(f(\bar{w}^i)', 1)'$ . Suppose  $\tau$  lies in the boundary of  $X$  and  $X = wF^-$ . Then  $x$  is a stationary point of  $\bar{f}$  on  $K$  if  $\tau$  lies in  $F^-$ . Otherwise, either  $\tau$  lies in  $wG^-$  with  $G$  a facet of  $F$  or  $\tau$  lies in  $wF^0$ . In the first case the path  $T$  can be continued in  $wG^-$  by pivoting  $((a_k)', 0)'$  into (4.3), where  $k$  is the unique index in  $I(G)$ , not in  $I(F)$ . In the latter case the path  $T$  can be continued in  $wF^0$  by pivoting  $(h', 0)'$  into (4.3). Now, suppose  $\tau$  lies in the boundary of  $X$  and  $X = wF^0$  or  $X = F^+$ . Then, when  $X = wF^0$ ,  $\tau$  lies either in  $wG^0$  for some facet  $G$  of  $F$  or in  $F^0$ ; and when  $X = F^+$ ,  $\tau$  lies either in  $G^+$  for some facet  $G$  of  $F$  or in  $F^0$ . When  $\tau$  is in  $wG^0$  or  $G^+$ , the path  $T$  can be continued by pivoting  $((a_k)', 0)'$  into (4.4), where  $k$  is the unique index in  $I(G)$ , not in  $I(F)$ . When  $\tau$  lies in  $F^0$ ,  $\tau$  is the facet of a unique  $t$ -simplex  $\bar{\sigma}$  in  $F^+$  if  $X = wF^0$  and in  $wF^0$  if  $X = F^+$ ; and the path  $T$  can be continued in  $\bar{\sigma}$  by making a pivoting step with  $(f(\bar{w})', 1)'$  in (4.4), where  $\bar{w}$  is the vertex of  $\bar{\sigma}$  opposite  $\tau$ .

We now consider the case that at an end point of solutions of (4.3) or (4.4) we have that  $\mu_k = 0$  for some  $k \in I(F)$ . Let  $G$  be the unique face of  $K$  such that  $I(G) = I(F) \setminus \{k\}$ . Then with  $x = \sum_i \lambda_i w^i$  we have that  $\bar{f}(x) = \sum_i \lambda_i f(w^i) = -\sum_{k \in I(G)} \mu_k a_k \in G^*$ . First, suppose that  $X = wF^-$ . If  $w \in G$ , then also  $x \in G$ , since  $F$  is a facet of  $G$ . Therefore,  $w \in G$  implies  $(x, -f(x)) \in G \times G^*$ , and hence  $x$  is a stationary point of  $\bar{f}$  on  $K$ . In case  $w \notin G$  or if  $X = wF^0$  or  $F^+$ , then  $\sigma$  is a facet of a unique  $(t+1)$ -dimensional simplex  $\bar{\sigma}$  in  $wG^-$ ,  $wG^0$ , or  $G^+$ , respectively; and the path can be continued in  $\bar{\sigma}$  by making a pivoting step with  $(f(\bar{w})', 1)'$  in (4.3) or (4.4), where  $\bar{w}$  is the vertex of  $\bar{\sigma}$  opposite  $\sigma$ .

Finally, we consider the case that in (4.4),  $\alpha = 0$  at an end point. If  $X = wF^0$  and  $w \notin F$ , then  $\sigma$  is a facet of a unique  $(t+1)$ -simplex  $\bar{\sigma}$  in  $wF^-$  and the path can be continued in  $wF^-$  by making a pivoting step with  $(f(\bar{w})', 1)'$  in (4.4), where  $\bar{w}$  is the



vertex of  $\bar{\sigma}$  opposite  $\sigma$ . If  $X = wF^0$  and  $w \in F$  or if  $X = F^+$ , then  $x = \sum_i \lambda_i w^i \in F$  and  $\bar{f}(x) = \sum_i \lambda_i f(w^i) = -\sum_{k \in I(F)} \mu_k a_k \in F^*$ , so that  $x$  is a stationary point of  $\bar{f}$  on  $K$ .

This completes the description of how to follow approximately the path  $S$  by making alternating pivoting and replacement steps for a sequence of adjacent simplices of varying dimension. When this sequence does not diverge and terminates with a simplex, it contains a stationary point  $\bar{x}$  of  $\bar{f}$  on  $K$ . This point  $\bar{x}$  is an approximate stationary point of  $f$  on  $K$ . To improve the accuracy of the approximation, if necessary, we can take a finer triangulation of  $K$  with the point  $\bar{x}$  as the new starting point  $w$  and apply the same procedure.

**5. Convergence condition.** In this section we state a condition under which the path  $S$  is bounded and therefore leads from  $w$  to a stationary point of  $f$  on  $K$ .

LEMMA 5.1. *Let  $(x, y)$  be a solution of the system*

$$(5.1) \quad g(x, y) = 0, \quad (x, y) \in F^+ \times (F^* + \langle h \rangle).$$

*If  $x$  is not a stationary point, then*

$$r'y > 0$$

*for any nonzero vector  $r$  in the cone  $C$  such that  $(a_i)'r = 0$  for all  $i \in I(F)$ .*

*Proof.* The point  $y$  in  $F^* + \langle h \rangle$  is equal to  $B\mu + \alpha h$  for some vector  $\mu \geq 0$  and number  $\alpha \geq 0$ , where  $B$  denotes the submatrix of  $A$  consisting of the column vectors  $a_i$  for  $i \in I(F)$ . Since  $x$  is not a stationary point,  $\alpha > 0$ . Then

$$r'y = r'(B\mu + \alpha h) = (B'r)' \mu + \alpha h'r = \alpha h'r > 0$$

by the choice of  $h$ .  $\square$

CONDITION 5.2. *There is a set  $U \subset R^n$  such that  $U \cap K$  is bounded and for each point  $x \in K \setminus U$  there is a nonzero vector  $\tilde{r}$  in  $C \cap \{r \in R^n \mid (a_i)'r = 0 \text{ if } (a_i)'x = b_i\}$  satisfying*

$$\tilde{r}'f(x) \geq 0.$$

LEMMA 5.3. *Under Condition 5.2 the path  $S$  does not diverge.*

*Proof.* Suppose the contrary. Then there is a solution  $(\tilde{x}, \tilde{y})$  of the system (4.1) such that  $\tilde{x} \in F^+ \setminus U$  for some face  $F$ , since the continuity of the function  $f$  requires the  $x$ -component to diverge. Therefore, by Lemma 5.1 and Condition 5.2, we see that  $\tilde{r}'(f(\tilde{x}) + \tilde{y}) > 0$  for some vector  $\tilde{r}$ , which contradicts the statement that  $(\tilde{x}, \tilde{y})$  is a solution of (4.1).  $\square$

**6. Stationary point problems on a Cartesian product of a polytope and a polyhedron.** We consider a stationary point problem defined on the Cartesian product of a polytope and a polyhedron. The product is again a polyhedron and the discussion of §§ 3, 4, and 5 could still be applied to this case. However, it will be quite useful to consider it separately because a lot of problems are defined on such product sets. Let

$$K_1 = \{x_1 \in R^{n_1} \mid A_1'x_1 \leq b_1\}$$

be a nonempty polytope and let

$$K_2 = \{x_2 \in R^{n_2} \mid A_2'x_2 \leq b_2\}$$

be a nonempty, convex, unbounded, and pointed polyhedron, with  $A_i$  an  $n_i \times m_i$  matrix and  $b_i$  an  $m_i$ -vector for  $i = 1, 2$ . We consider the stationary point problem for a continuous function  $f$  from  $K_1 \times K_2$  to  $R^{n_1} \times R^{n_2}$ . We denote  $f(x)$  by  $f(x) = (f_1(x_1, x_2), f_2(x_1, x_2))$ . Then  $(x_1, x_2) \in K_1 \times K_2$  is a stationary point of  $f$  on  $K_1 \times K_2$  if

$$(z_1 - x_1)'f_1(x_1, x_2) + (z_2 - x_2)'f_2(x_1, x_2) \geq 0$$

for any point  $(z_1, z_2) \in K_1 \times K_2$ .



In the same way as in the preceding sections, we will construct a GPDM by introducing an artificial hyperplane and corresponding halfspaces defined by

$$H^\pi = \{(x_1, x_2) \in R^{n_1+n_2} \mid h_2' x_2 \rho h_0\},$$

where  $\pi$  is  $-$ ,  $0$ , and  $+$  when  $\rho$  is  $\leq$ ,  $=$ , and  $\geq$ , respectively.  $h_2 = -A_2 \gamma$  for some fixed positive vector  $\gamma$ , and  $h_0 > 0$  is chosen such that the interior of the halfspace  $H^-$  contains all vertices of  $K_1 \times K_2$  as well as the starting point  $w = (w_1, w_2)$ . Note that  $h_2' r_2 > 0$  for any nonzero vector  $r_2$  in the set

$$C_2 = \{r_2 \in R^{n_2} \mid A_2' r_2 \leq 0\}$$

of directions of rays of  $K_2$ , which we have seen is a pointed cone. It is clear that a face  $F$  of  $K_1 \times K_2$  is itself a Cartesian product of faces of  $K_1$  and  $K_2$ , which we will denote by  $F_1$  and  $F_2$ , respectively. Let

$$H_2^\pi = \{x_2 \in R^{n_2} \mid h_2' x_2 \rho h_0\},$$

where  $\pi$  is  $-$ ,  $0$ , and  $+$  when  $\rho$  is  $\leq$ ,  $=$ , and  $\geq$ , respectively. We define

$$F_2^\pi = F_2 \cap H_2^\pi \quad \text{for } \pi = -, 0, \text{ and } +.$$

Then

$$F^\pi = F_1 \times F_2^\pi \quad \text{for } \pi = -, 0, \text{ and } +.$$

It is also clear that the normal cone  $F^*$  corresponding to a face  $F$  of  $K$  is given by

$$F^* = F_1^* \times F_2^*,$$

where  $F_1^*$  and  $F_2^*$  are defined with respect to  $K_1 \subset R^{n_1}$  and  $K_2 \subset R^{n_2}$ , respectively. Thus, with the dual operator  $d$  defined as follows, we obtain a GPDM:

$$\begin{aligned} (w(F_1 \times F_2^-))^d &= F_1^* \times F_2^* && \text{if } w \notin F_1 \times F_2 < K; \\ (w(F_1 \times F_2^0))^d &= F_1^* \times (F_2^* + \langle h_2 \rangle) && \text{if } F_2 \text{ is an unbounded face of } K_2; \\ (F_1 \times F_2^+)^d &= F_1^* \times (F_2^* + \langle h_2 \rangle) && \text{if } F_2 \text{ is an unbounded face of } K_2; \\ (F_1 \times F_2^-)^d &= \emptyset && \text{if } w \notin F_1 \times F_2 < K; \\ (F_1 \times F_2^0)^d &= \emptyset && \text{if } F_2 \text{ is an unbounded face of } K_2; \\ (\{w\})^d &= \emptyset. \end{aligned}$$

The collection  $\mathcal{M}$  of cells, each cell being the Cartesian product of a primal cell and its dual, is clearly a subdivided  $(n_1 + n_2 + 1)$ -manifold. The boundary  $\partial \mathcal{M}$  contains  $\{(w_1, w_2)\} \times (R^{n_1} \times R^{n_2} \setminus \{(w_1, w_2)\}^*)$ . It also contains  $(F_1 \times F_2) \times (F_1^* \times F_2^*)$  for all faces  $F_1$  of  $K_1$  and for all faces  $F_2$  of  $K_2$  when  $(w_1, w_2)$  is not a vertex of  $K$ ; and when  $(w_1, w_2)$  is a vertex of  $K$ , it contains  $(F_1 \times F_2) \times (F_1^* \times F_2^*)$  for all faces  $F_1$  of  $K_1$  not equal to  $\{w_1\}$  and for all faces  $F_2$  of  $K_2$  not equal to  $\{w_2\}$ . Therefore, when the starting point  $w = (w_1, w_2)$  is not a stationary point, the point  $(x_1, x_2, y_1, y_2) = (w_1, w_2, -f_1(w_1, w_2), -f_2(w_1, w_2))$  lies in the boundary  $|\partial \mathcal{M}|$  of  $|\mathcal{M}|$  and under the regular value assumption there is a path leading from it to either a stationary point or to infinity. Thus, in exactly the same way as in the preceding sections, the problem is now reduced to tracing the path  $S$  of solutions to the system

$$(f_1(x_1, x_2), f_2(x_1, x_2)) + (y_1, y_2) = 0, \quad (x_1, x_2, y_1, y_2) \in |\mathcal{M}|.$$

The remarkable feature of this path is shown in the following lemma, where

$$S_x = \{x = (x_1, x_2) \mid (x_1, x_2, y_1, y_2) \in S \text{ for some } (y_1, y_2) \in R^{n_1+n_2}\}.$$



LEMMA 6.1. If  $(\tilde{x}_1, \tilde{x}_2) \in S_x \cap H^+$ , then  $\tilde{x}_1$  is a stationary point for  $f_1(\cdot, \tilde{x}_2)$  on  $K_1$ , i.e.,  $\tilde{x}_1' f_1(\tilde{x}_1, \tilde{x}_2) \leq x_1' f_1(\tilde{x}_1, \tilde{x}_2)$  for all  $x_1 \in K_1$ .

*Proof.* Since  $(\tilde{x}_1, \tilde{x}_2) \in H^+$ , it is in  $F_1 \times F_2^+$  for some face  $F_1$  of  $K_1$  and some unbounded face  $F_2$  of  $K_2$ . By the construction of the GPDM,

$$(-f_1(\tilde{x}_1, \tilde{x}_2), -f_2(\tilde{x}_1, \tilde{x}_2)) \in F_1^* \times (F_2^* + \langle h_2 \rangle).$$

This means that  $\tilde{x}_1$  is a stationary point for  $f_1(\cdot, \tilde{x}_2)$  on  $K_1$ .  $\square$

LEMMA 6.2. Let  $(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$  be a point of  $S$ . Suppose that  $(\tilde{x}_1, \tilde{x}_2)$  is not a stationary point and  $\tilde{x}_2$  lies in  $H_2^+$ . Then

$$r_2' \tilde{y}_2 > 0$$

for any nonzero vector  $r_2$  in the cone  $C_2$  such that  $(a_{2i})' r_2 = 0$  whenever  $(a_{2i})' \tilde{x}_2 = b_{2i}$ , where  $a_{2i}$  is the  $i$ th column of  $A_2$  and  $b_{2i}$  is the  $i$ th component of  $b_2$ .

*Proof.* Let  $B_2$  be the submatrix of  $A_2$  consisting of the columns  $a_{2i}$  such that  $(a_{2i})' \tilde{x}_2 = b_{2i}$ . Then  $\tilde{y}_2 = B_2 \mu + \alpha h_2$  for some vector  $\mu \geq 0$  and real number  $\alpha \geq 0$ . Since  $(\tilde{x}_1, \tilde{x}_2)$  is in  $H^+$  and is not a stationary point, we have  $\alpha > 0$  by Lemma 6.1. Therefore,

$$r_2' \tilde{y}_2 = r_2' (B_2 \mu + \alpha h_2) = \alpha h_2' r_2 > 0. \quad \square$$

CONDITION 6.3. There is a set  $U_2 \subset R^{n_2}$  such that  $U_2 \cap K_2$  is bounded and for each point  $\tilde{x}_2 \in K_2 \setminus U_2$  one of the following conditions holds:

- (1) there is no stationary point for  $f_1(\cdot, \tilde{x}_2)$  on  $K_1$ ,
- (2) for each point  $x_1 \in K_1$ , there is a nonzero vector  $\tilde{r}_2$  in  $C_2 \cap \{r_2 \in R^{n_2} \mid (a_{2i})' r_2 = 0 \text{ if } (a_{2i})' \tilde{x}_2 = b_{2i}\}$  such that  $\tilde{r}_2' f_2(x_1, \tilde{x}_2) \geq 0$ .

THEOREM 6.4. Under Condition 6.3 the path  $S$  does not diverge.

*Proof.* Suppose the contrary. Then there is a point  $(\tilde{x}_1, \tilde{x}_2) \in S_x \cap H^+$  such that  $\tilde{x}_2 \notin U_2$ . By Lemma 6.1,  $\tilde{x}_1$  is a stationary point for  $f_1(\cdot, \tilde{x}_2)$  on  $K_1$ . Therefore, condition (2) must be satisfied at this point, so that for some nonzero vector  $\tilde{r}_2$  in  $C_2 \cap \{r_2 \in R^{n_2} \mid (a_{2i})' r_2 = 0 \text{ if } (a_{2i})' \tilde{x}_2 = b_{2i}\}$  we must have

$$\tilde{r}_2' f_2(\tilde{x}_1, \tilde{x}_2) \geq 0.$$

On the other hand, we have seen in Lemma 6.2 that

$$\tilde{r}_2' f_2(\tilde{x}_1, \tilde{x}_2) = \tilde{r}_2' (-\tilde{y}_2) < 0.$$

This is a contradiction.  $\square$

**7. Linear stationary point problems.** In this section, we consider a special but important case where the function  $f$  from  $K$  to  $R^n$  is an affine function, i.e.,  $f(x) = Qx + q$ , where  $Q$  is an  $n \times n$  matrix and  $q$  is an  $n$ -vector. For simplicity of notation we confine ourselves to the linear stationary point problem defined on a polyhedron instead of the product of a polytope and a polyhedron. As for complementary pivoting algorithms for solving a linear complementarity problem, we show that if the matrix  $Q$  is copositive plus on the polyhedral cone  $C$  and the problem has a stationary point, the path does not go to infinity and consequently leads to one of the stationary points.

DEFINITION 7.1. The matrix  $Q$  is copositive plus on  $C$  if

- (1)  $r' Q r \geq 0$  for any  $r \in C$ ,
- (2)  $(Q + Q')r = 0$  if  $r \in C$  and  $r' Q r = 0$ .

LEMMA 7.2. There exists no point  $x \in K$  such that  $Qx + q = -A\mu$  for some vector  $\mu \geq 0$  if and only if there is a  $(v, u) \in R^n \times R^m$  such that  $v \in C$ ,  $Q'v = Au$ ,  $b'u + q'v < 0$ , and  $u \geq 0$ .



*Proof.* There exists no point  $x$  in  $K$  satisfying  $Qx + q = -A\mu$  for some  $\mu \geq 0$  if and only if the system

$$(7.1) \quad A'(x_1 - x_2) \leq b, \quad Q(x_1 - x_2) + q = -A\mu, \quad x_1, x_2, \mu \geq 0$$

is not solvable. By Farkas' Alternative Theorem, we have an equivalent statement to (7.1): the following system:

$$(7.2) \quad Q'v - Au = 0, \quad A'v \leq 0, \quad u \geq 0, \quad b'u + q'v < 0$$

is solvable. This means the existence of a point  $v$  in  $C$  such that  $Q'v = Au$  and  $b'u + q'v < 0$  for some  $u \geq 0$ .  $\square$

LEMMA 7.3. *Let  $Q$  be copositive plus on  $C$ . If the path  $S$  is unbounded and does not contain a point which provides a stationary point, then there are no stationary points.*

*Proof.* Suppose  $S$  is unbounded, then there are  $(x, y) \in S$  and  $(\bar{x}, \bar{y}) \neq 0$  such that  $(x, y) + \beta(\bar{x}, \bar{y}) \in S$  for any  $\beta \geq 0$ . Then

$$(7.3) \quad \bar{y} + Q\bar{x} = 0.$$

Moreover, as  $\beta$  increases,  $(x, y) + \beta(\bar{x}, \bar{y})$  will be entirely contained in a cell  $F^+ \times (F^* + \langle h \rangle)$  for some face  $F \leq K$ . Here note that  $\bar{x} \neq 0$  because the contrary would yield  $(\bar{x}, \bar{y}) = 0$ . Then we have

$$x \in F^+$$

$$y = y' + \lambda h \quad \text{for some } y' \in F^* \text{ and some } \lambda \geq 0,$$

and

$$\bar{x} \in C \cap \{r \in R^n \mid (a_i)'r = 0 \text{ if } (a_i)'x = b_i\}$$

$$\bar{y} = \bar{y}' + \mu h \quad \text{for some } \bar{y}' \in F^* \text{ and some } \mu \geq 0.$$

Therefore, we have

$$\bar{x}'Q\bar{x} = \bar{x}'(-\bar{y}) = \bar{x}'(-\bar{y}' - \mu h) = -\mu\bar{x}'h.$$

Suppose  $\mu > 0$ . By the choice of  $h$  and since  $\bar{x} \in C$ , we have  $\mu\bar{x}'h > 0$ , which contradicts that  $Q$  is copositive plus on  $C$ . Therefore,  $\mu = 0$  and  $\bar{x}'Q\bar{x} = 0$ . If  $\lambda = 0$ , then  $y = y' \in F^*$ . This means that the point  $x$  is a stationary point. Since we have assumed that  $S$  does not contain such a point, we see that  $\lambda > 0$ . Since  $\bar{x}'Q\bar{x} = 0$  implies  $(Q + Q')\bar{x} = 0$ , we have

$$Q'\bar{x} = -Q\bar{x} = \bar{y} = \bar{y}' + \mu h = \bar{y}' \in F^*.$$

In other words, there is some vector  $u$  satisfying

$$Q'\bar{x} = Au,$$

$$(7.4) \quad \begin{aligned} u_i &\geq 0 \quad \text{for } i \in I(F), \\ u_i &= 0 \quad \text{for } i \notin I(F). \end{aligned}$$

We also have

$$(7.5) \quad \bar{x}'(-y) = \bar{x}'(Qx + q) = x'Q'\bar{x} + q'\bar{x} = x'(-Q\bar{x}) + q'\bar{x} = x'\bar{y} + q'\bar{x}.$$

On the other hand, since  $\bar{x} \in C \cap \{r \in R^n \mid (a_i)'r = 0 \text{ if } (a_i)'x = b_i\}$  and  $y' \in F^*$ ,

$$(7.6) \quad \bar{x}'(-y) = \bar{x}'(-y' - \lambda h) = -\bar{x}'y' - \lambda\bar{x}'h = -\lambda\bar{x}'h < 0.$$



From (7.5) and (7.6) we have  $x'\bar{y} + q'\bar{x} < 0$ . Since  $x \in F^+$ , we also have that  $A'x + s = b$  for some slack variable vector  $s$  satisfying

$$s_i \geq 0 \quad \text{for } i \notin I(F),$$

$$s_i = 0 \quad \text{for } i \in I(F).$$

Then

$$(7.7) \quad \begin{aligned} b'u + q'\bar{x} &< b'u - x'\bar{y} = (A'x + s)'u - x'(-Q\bar{x}) \\ &= x'Au + s'u - x'(Q'\bar{x}) = x'(Au - Q'\bar{x}) + s'u = 0. \end{aligned}$$

From (7.4), (7.7), and Lemma 7.2, we conclude that there are no stationary points.  $\square$

The algorithm for tracing the piecewise linear path  $S$ , being linear on each cell of  $\mathcal{M}$ , is quite similar to that proposed in Yamamoto [8] for solving linear stationary point problems on polytopes. We will only give an outline here. Suppose we are at a point  $(x, y)$  on the path, i.e.,

$$(7.8) \quad Qx + q + y = 0, \quad (x, y) \in X \times X^d,$$

for some cell  $X \times X^d$  of  $\mathcal{M}$ . By the decomposition theorem of a polyhedron, each point of a polyhedron is a sum of two points: a convex combination of vertices of the polyhedron and a nonnegative combination of directions of extreme rays. Let  $U$  and  $R$  be the sets of vertices and extreme rays of  $X$ , respectively. Then a point  $x \in X$  is written as

$$\begin{aligned} x &= \sum_{u \in U} \lambda_u u + \sum_{r \in R} \alpha_r r, \\ \sum_{u \in U} \lambda_u &= 1, \\ \lambda_u &\geq 0, \quad \alpha_r \geq 0. \end{aligned}$$

On the other hand,  $X^d$  is the cone generated by coefficient vectors  $a_i$  of binding constraints of the face corresponding to  $X$  and the vector  $h$ . Then a point  $y \in X^d$  is written as

$$y = \sum \mu_i a_i, \quad \mu_i \geq 0$$

if we denote  $h$  by  $a_0$ . Therefore, (7.8) has a solution if and only if the system

$$(7.9) \quad \begin{aligned} \sum \lambda_u \begin{bmatrix} Qu \\ 1 \end{bmatrix} + \sum \alpha_r \begin{bmatrix} Qr \\ 0 \end{bmatrix} + \sum \mu_i \begin{bmatrix} a_i \\ 0 \end{bmatrix} &= \begin{bmatrix} -q \\ 1 \end{bmatrix}, \\ \lambda_u &\geq 0, \quad \alpha_r \geq 0, \quad \mu_i \geq 0 \end{aligned}$$

has a solution  $(\lambda, \alpha, \mu)$ . It should be noted here that a vertex of  $X$  is either the starting point  $w$  or a vertex of some face of  $K^-$  corresponding to  $X$  and that an extreme ray of  $X$  is also an extreme ray of some face of  $K$ . More precisely,

$$U = \{w\} \cup \{\text{vertices of } F^-\}, \quad R = \emptyset \quad \text{when } X = wF^-,$$

$$U = \{w\} \cup \{\text{vertices of } F^0\}, \quad R = \emptyset \quad \text{when } X = wF^0,$$

$$U = \{\text{vertices of } F^0\}, \quad R = \{\text{extreme rays of } F\} \quad \text{when } X = F^+.$$

In every case a vertex or an extreme ray can be generated in need when we keep in storage the index set of binding constraints, including  $H^0 = \{x \in R^n \mid h'x = h_0\}$ , determining the face  $F$ .



Suppose we are at an end point of the line segment or halfline of the path within  $X \times X^d$ . Since the path is linear within  $X \times X^d$ , an appropriate choice of the objective function  $c_x x + c_y y$  makes the current end point the unique maximal solution of the linear program:

$$\begin{aligned} \max \quad & c_x x + c_y y, \\ \text{s.t.} \quad & x = \sum \lambda_u u + \sum \alpha_r r, \\ & y = \sum \mu_i a_i, \\ & \sum \lambda_u \begin{bmatrix} Qu \\ 1 \end{bmatrix} + \sum \alpha_r \begin{bmatrix} Qr \\ 0 \end{bmatrix} + \sum \mu_i \begin{bmatrix} a_i \\ 0 \end{bmatrix} = \begin{bmatrix} -q \\ 1 \end{bmatrix}, \\ & \lambda_u \geq 0, \quad \alpha_r \geq 0, \quad \mu_i \geq 0. \end{aligned}$$

In fact, the outward normal vector at the point  $(x, y)$  to  $X \times X^d$  may serve as  $(c_x, c_y)$ . Then the other end point, when the path within  $X \times X^d$  is a line segment, or the diverging direction, when it is a halfline, can be found by solving the following linear minimization program:

$$\begin{aligned} \min \quad & c_x x + c_y y, \\ \text{s.t.} \quad & x = \sum \lambda_u u + \sum \alpha_r r, \\ & y = \sum \mu_i a_i, \\ & \sum \lambda_u \begin{bmatrix} Qu \\ 1 \end{bmatrix} + \sum \alpha_r \begin{bmatrix} Qr \\ 0 \end{bmatrix} + \sum \mu_i \begin{bmatrix} a_i \\ 0 \end{bmatrix} = \begin{bmatrix} -q \\ 1 \end{bmatrix}, \\ & \lambda_u \geq 0, \quad \alpha_r \geq 0, \quad \mu_i \geq 0. \end{aligned}$$

From this we see that solving the problem is a typical application of the Dantzig-Wolfe decomposition principle for large structured linear programs. By solving a sequence of these problems we can trace the path and finally, after a finite number of iterations, we meet with an end point of the path or find that the path goes to infinity.

#### REFERENCES

- [1] Y. DAI AND Y. YAMAMOTO, *The path following algorithm for stationary point problems on polyhedral cones*, J. Oper. Res. Soc. Japan, 32 (1989), pp. 286-309.
- [2] M. KOJIMA, *An introduction to variable dimension algorithms for solving systems of equations*, in Numerical Solution of Nonlinear Equations, Springer-Verlag, Berlin, New York, 1981, pp. 199-237.
- [3] S. KARADIMIAN, *Generalized complementarity problem*, J. Optim. Theory Appl., 8 (1971), pp. 161-168.
- [4] M. KOJIMA AND Y. YAMAMOTO, *Variable dimension algorithms: Basic theory, interpretations and extensions of some existing methods*, Math. Programming, 24 (1982), pp. 177-215.
- [5] R. SAIGAL, *Extension of the generalized complementarity problem*, Math. Oper. Res., 1 (1976), pp. 260-266.
- [6] J. STOER AND C. WITZGALL, *Convexity and Optimization in Finite Dimensions I*, Springer-Verlag, Berlin, New York, 1978.
- [7] A. J. J. TALMAN AND Y. YAMAMOTO, *A simplicial algorithm for stationary point problems*, Math. Oper. Res., 14 (1989), pp. 383-399.
- [8] Y. YAMAMOTO, *A path following algorithm for stationary point problems*, J. Oper. Res. Soc. Japan, 30 (1987), pp. 181-198.
- [9] ———, *Stationary point problems and a path-following algorithm*, in Proc. 8th Mathematical Programming Symposium, Hiroshima, Japan, 1987, pp. 153-170.
- [10] ———, *Fixed point algorithms for stationary point problems*, Mathematical Programming, Recent Developments and Applications, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1989, pp. 283-307.